

Cox-Ingersoll-Ross and Black-Derman-Toy Models

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Recap: The Vasicek Model

In the Vasicek model, the short term interest rate follows

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$$

where we interpreted the various terms to mean

- dr_t the change in the short term interest rate.
- κ : the speed of mean reversion. It approximately equals $1 - b$ where b is the first order autocorrelation.
- θ : the average interest rate
- σ : the volatility of the short rate
- dW_t : a random term so that $E(dW) = 0$.

Behavior of Interest Rates in Vasicek's Model: Expectations

Expected changes in interest rates are given by

$$E(dr_t) = \kappa(\theta - r_t)dt.$$

We call this the *drift*.

We have that

- $E(dr_t) > 0$ when $r_t < \theta$
- $E(dr_t) < 0$ when $r_t > \theta$

- When $\kappa = 0$ interest rates are *random walk* and *they do not mean revert*. In this case, there does not exist a long run average interest rate.
- When κ is small, interest rates exhibit *high persistence* and they tend to *mean revert* slowly.
- When κ is large, interest rates mean revert quickly.

Suppose we draw the random shocks, e_t out of a hat for $t = 1, \dots, T$. All the numbers in the hat are normally distributed with mean zero and variance one.

We can now use these numbers to simulate what would happen to future interest rates. We use the equation

$$r_t = r_{t-1} + \kappa(\theta - r_{t-1})dt + \sigma\sqrt{dt}e_t \quad (1)$$

for some dt .

In the last lecture I argued that $\kappa = 0.2$, $\theta = 0.05$ and $\sigma = 0.018$ represent reasonable values describing the year-to-year behavior of US interest rates.

- Get normally distributed random numbers using $NORMSINV(RAND())$
- Use equation (1) to compute all future interest rates as functions of κ, θ, σ and the time step dt .

See that

- When interest rates are high (low), the subsequent rates tend to decrease (increase) toward the long run average rate, θ .
- When κ is high, rates mean revert more quickly
- Interest rates can become negative in the Vasicek model

One of the features of the Vasicek model is that interest rate changes have constant volatility, σ , no matter what happens in the economy.

It can be useful to relax this assumption for two reasons

- We have empirical evidence suggesting that interest rate changes are *more* volatile when the level of interest rates are high
- As we will show, interest rates will remain positive..

The model is

$$dr_t = \kappa(\theta - r_t)dt + \sigma^* \sqrt{r_t} dW_t$$

Thus, the CIR model has the same *drift* term as the Vasicek model, but the volatility is

$$\text{Std}(dr_t) = \sigma^* \sqrt{r_t}$$

This is an example of a *conditional* volatility: when the short rate r_t is high the volatility of interest rate changes is high and vice versa.

The parameter σ^* no longer represents the volatility of interest rate changes. It is just a parameter...

Choosing σ^*

Last lecture we argued that the volatility of short rates changes are some 180 basis points per year. In the CIR model, volatility is time varying.

It is reasonable therefore to choose σ^* such that the average volatility under the CIR model is approximately equal to 180 basis point (0.018).

We can do this by solving

$$0.018^2 = E(\text{Var}(dr_t)) = \sigma^{*2} E(r_t) = (\sigma^*)^2 0.05$$

or

$$\sigma^* = \sqrt{\frac{0.018^2}{0.05}} \approx 0.08$$

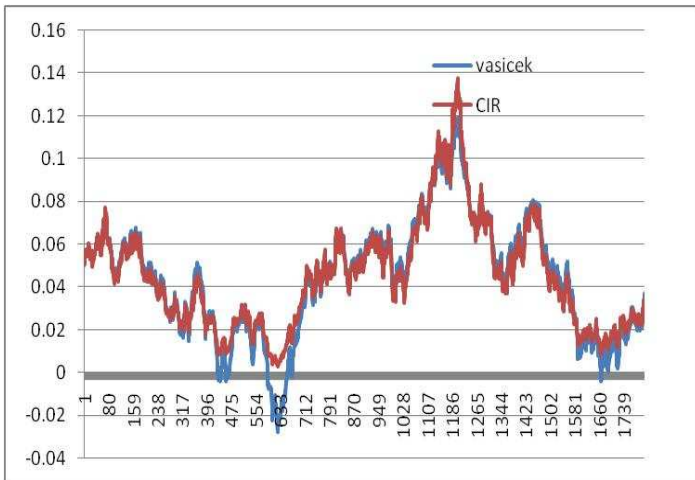


Figure: Simulated Vasicek and CIR interest rates. Shocks are the same.

Notice that the simulated CIR path is *positive* while becoming negative for the Vasicek model.

This is not a coincidence: In theory, all CIR paths should remain positive....

Our excel sheet for simulation may produce negative values still because our simulations are approximations to the continuous time model in CIR....

Prices of zero coupons with maturity T are given by

$$P(T) = A(T)e^{-B(T)r_0} \quad (2)$$

where

$$A(T) = \left[\frac{2he^{(\kappa+h)T/2}}{2h + (\kappa + h)(e^{hT} - 1)} \right]^{\frac{2\kappa\theta}{\sigma^2}} \quad (3)$$

$$B(T) = \frac{2(e^{hT} - 1)}{2h + (\kappa + h)(e^{hT} - 1)} \quad (4)$$

$$h = \sqrt{\kappa^2 + 2\sigma^2} \quad (5)$$

Trees that fit the current term structure

Consider a world in which you observed the following zero coupon bond ytm's:

maturity	1	2	3	4	5
ytm	0.1	0.11	0.12	0.12.5	0.13
price	0.901	0.812	0.712	0.624	0.543

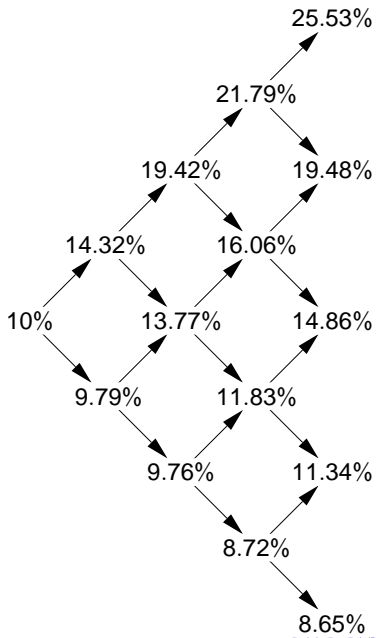
How can we build a tree such that it prices these zero coupon bonds correctly?

Yes, and there are several methods available for constructing such trees:

- Ho-Lee
- Hull White
- Black-Derman-Toy
- Black-Karasinsky

These models make different assumptions about the behavior of interest rates. We will consider first an example from Black-Derman-Toy.

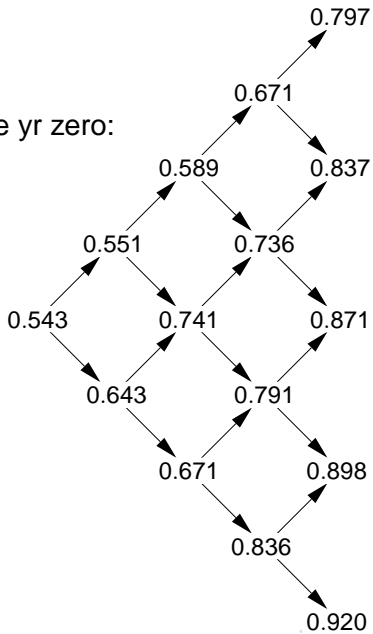
BDT tree:



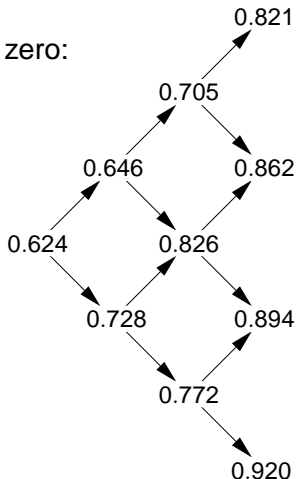
How do we know that this tree is consistent with the current term structure?

Let's price a 5 year zero in the tree. It is...

Price tree for the five yr zero:



Similarly, the four yr zero:



Thus, we verify the prices of both the 5 and 4 year zeros. We can do the same exercise for the 3 and 2 of course....

Properties of the BDT model

- Model has 50/50 risk neutral probabilities of up/down moves
- The short term interest rate is *log-normally* distributed
- Rates are always positive
- Model assumes that you can estimate future short rate volatility. We need a volatility model to make such forecasts (i.e., GARCH)

Lets look at the ratio of the up/down states at some random time/state, for example the up/up/up and up/up/down, in which case the rates are 25.53 and 19.48. The ratio is

$$\frac{25.53}{19.48} = 1.31$$

Going one step down, we have

$$\frac{19.48}{14.86} = 1.31$$

... further ...

$$\frac{14.86}{11.34} = 1.31$$

...in other words.. the ratio of the up/down states are the same for each state.

At time step 4 we find the ratio to be 1.35, at time 3 we find 1.41, at time 2 we find 1.46.

A measure of volatility in the BDT model is

$$\sigma(t) = \frac{1}{2} \ln r^u(t)/r^d(t)$$

Since the ratio of the up/down rates are the same irrespective of the level of interest rates, $\sigma(t)$ does not depend on the *level* of interest rates, as in CIR.

$\sigma(t)$ is however *maturity specific*. This means that the BDT model allows for the volatility of future short rates to depend on time.

Example: Suppose we were to believe that future short rates will be less volatile than today, then we could specify a decreasing volatility structure, for example

$$\sigma(t) = [0.2, 0.16, 0.14, 0.13, 0.12\dots]$$

would suggest that we expect volatility to decrease from its current level of 200 basis point/ year to 120 basis point in the next 5 years...

On the other hand, we could specify

$$\sigma(t) = [0.05, 0.07, 0.09, 0.1, 0.11\dots]$$

to suggest an increasing volatility structure.

The downward (upward) sloping volatility scheme would be practically relevant when financial markets are very volatile (not volatile).

Concluding Remarks: CIR

CIR model advantages:

- explicit formulas for term structure, option prices, etc
- based on a coherent statistical model for behavior of interest rates

CIR disadvantages:

- Statistical model too simple
- term structure not flexible enough to fit observed term structures
- all bond prices are 100% correlated (not empirically true)

Concluding Remarks: BDT

BDT model advantages:

- always fits term structure
- can fit reasonable forecasts of volatility

BDT disadvantages:

- underlying statistical model of interest rate behavior is possibly not verifiable
- all bond prices are 100% correlated (not empirically true). This is true for any one factor model of the short rate.