

Forward Contracts

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A forward contract on some asset is an agreement today to purchase the asset at an agreed upon price (the forward price) today, for delivery some time in the future (settlement date).

Notation: (more detailed than Tuckman)

- $P(t, T)$: The time t price of a T maturity zero. Remember that $P(t, T)$ is the same as the discount function $d(t, T)$.
- $F(0, S, T) = F$: The time 0 forward price for a T maturity zero with settlement S ($S < T$).

Cash flows from a forward contract

Assume initial date is time 0:

- At time 0 : No cash flows. Enter contract. Agree to pay $F(0, S, T)$.
- At time S : $-F$ (pay the forward price, receive the bond)
- At time T : Zero matures and we receive \$ 1.

Pricing a forward on a zero

Consider the following strategy at time 0:

- Buy the forward on a T maturity zero with settlement S .
- Sell one T zero short. Collect $P(0, T)$.
- Invest proceeds in the S zero.

We invest in the S zero for a total amount of $P(0, T)$.

The price of the S zero is $P(0, S)$, so the total number of bonds we buy is

$$n = \frac{P(0, T)}{P(0, S)}$$

such that $n \times P(0, S) = P(0, T)$ - the total proceeds from the short sale.

We now have the following cash flows:

At time 0: 0

At time S : - pay forward $-F(0, S, T)$

- collect \$ 1 coupon for every n S bonds.

Total cash flow is

$$\frac{P(0, T)}{P(0, S)} - F(0, S, T)$$

At time T : 0

Since the time S profit is riskless, it must be that

$$F(0, S, T) = \frac{P(0, T)}{P(0, S)} \quad (1)$$

is the arbitrage free price.

Coupon Bonds

Lets look at a coupon bond with a coupon payment $c/2$ taking place at some date T_c . Let's first assume that the coupon date is before the expiration of the forward. We assume for simplicity that the bond only pays coupon once before the forward expiration. Let $P_c(t)$ denote the time t (dirty) price of a maturity coupon bond. (Here, the maturity date of the coupon bond is not important and we omit it from the notation)

By entering into the forward contract, we receive the bond at time S but we do not receive the first coupon payment.

Consider the following strategy at time 0:

- Buy the forward on the coupon bond.
- Sell the coupon bond short. Collect $P_c(0)$.
- Invest

$$\frac{c}{2}P(0, T_c)$$

in the T_c maturity zero.

- Invest the remaining proceeds from short sale in S zeros. This gives us

$$n = \frac{P_c(0) - \frac{c}{2}P(0, T_c)}{P(0, S)}$$

units of S zeros.

Cash flows:

At time 0: 0

At time T_c : Receive $c/2$. Use to pay coupon holders. zero cash flow.

At time S . Pay forward price,

$$-F$$

Use bond to cover short. Long position in S zeros pays n . The total cash flow is $n - F$.

So it must be that the arbitrage free forward price is

$$F = \frac{P_c(0) - \frac{c}{2}P(0, T_c)}{P(0, S)}. \quad (2)$$

Generalization:

As you may imagine, the price of a forward on a coupon bond with multiple future coupon payments at times t_i $i = 1, \dots, n$ is just

$$F = \frac{P_c(0) - \frac{c}{2} \sum_{i=1}^n P(0, T_i)}{P(0, S)} \quad (3)$$

Using Repos

Suppose we can enter into a repo contract with any maturity t . If $r_p(t)$ is the repo rate for a time 0 to t investment, then it must be that

$$P(0, t) = \frac{1}{1 + r_p(t) \frac{t}{360}} \quad (4)$$

Otherwise, we could just arbitrage directly between the repo and underlying bond market.

Now we can take any of the preceding parity formulas and exchange the zero coupon bond prices with repo-based discount factors. For example, let's take the forward on a zero for which we saw that the parity formula was

$$F(0, S, T) = \frac{P(0, T)}{P(0, S)}$$

We get

$$F(0, S, T) = P(0, T)(1 + r_p(S)S/360)$$

For the coupon bond with a single coupon we get the following equivalent formulas

$$F = \frac{P_c(0) - \frac{c}{2}P(0, T_c)}{P(0, S)} \quad (5)$$

$$= (P_c(0) - \frac{c}{2}P(0, T_c))(1 + r_p(S)S/360) \quad (6)$$

$$= \left[P_c(0) - \frac{c/2}{1 + r_p(T_c)T_c/360} \right] (1 + r_p(S)S/360) \quad (7)$$

Eqn. (??) compares to equation 16.11 in Tuckman's book. Notice that in his book it is assumed that the repo rates are independent of the maturity of the investment. This is incorrect, and while it is inconsistent with market practice, but more importantly because it introduces arbitrage.

In practice, you are better off using formulas such as (??) to compute the value of forward contracts because we do not need to assume the existence of long dated repo's.

Later: Futures contracts.

Note

- Forward and futures prices differ when interest rates are stochastic.
- Actual bond futures are typically not written on a specific bond. Rather, the issuer has the *option* to deliver any bond in a basket (called cheapest to deliver option). This makes bond/note futures complicated.

Expectations Hypothesis

An old theory of forward rates suggests that they are expected future interest rates. Let f_T denote a T maturity forward rate

$$f_T = 1/F_T - 1$$

The EH suggests that

$$E(r_{T-1:T}) = f_T$$

I.e., the expected future short rate between $T - 1$ and T equals the forward rate.

THE EXPECTATIONS HYPOTHESIS IS LARGELY INCONSISTENT WITH MODERN FINANCE

To see why, consider the graph on the next page....

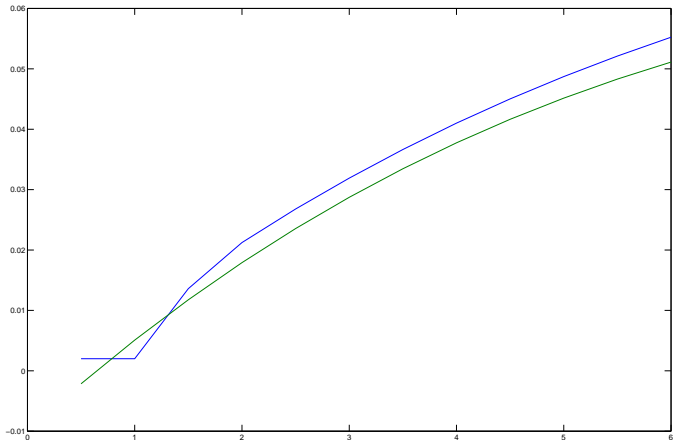


Figure: Expected Rates vs. forward rates in the calibrated BDT tree (March 2010)

What's wrong with the EH

We can show that

$$F_T = E^Q [B(S, T)]$$

- But

$$F_T = E^Q [B(S, T)] \geq \frac{1}{(1 + E^Q(r_{S:T})^{T-S})}$$

The inequality follows from the so-called Jensen inequality which stipulates that

$$E(f(x)) \geq f(E(x))$$

for f convex.

- The EH in its traditional form does not recognize the difference between risk neutral expectations and objective expectations