

Bond Arbitrage

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In baby finance, we compute the present values of cash flows received in $1, 2, \dots, T$ years from now. The value of a security that pays a cash flow $C(t)$, $t = 1, \dots, T$ is

$$V_0 = \sum_{t=1}^T C(t)d(t) \quad (1)$$

where $d(t)$ is a *discount function*. We discuss this in more detail below, but if interest rates are constant, then

$$d(t) = (1 + r)^{-t} \quad (2)$$

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Let us instead assume that a semi-annual coupon bond pays its first coupon in 30 days from now. In this case, the first coupon is in $1/12=0.083333$ years from now, the second is in 0.58333 years from now, the third in 1.08333 years from now and so on.

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Lets label the n coupon dates by t_1, t_2, \dots, t_n . The present value is

$$V_0 = \sum_{i=1}^n C(t_i)d(t_i). \quad (3)$$

In order to use this formula, we need to figure out what the discount function $d(t)$ is at the specific times $t_1 = 0.0833, t_2 = .5833\dots$

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The Discount Function

The discount function, $d(t)$, denotes today's value of \$1 paid t years from now.

Suppose, for example, that $d(0.5) = 0.98$ then the value of one dollar paid six months from now is

$$0.98 \times 1 = 0.98,$$

while the value of \$4 paid six months from now is

$$0.98 \times 4 = 3.92.$$

Likewise the value of a \$104 paid in one year from now is (assume $d(1) = 0.95$),

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Example: A 8% semi-annual coupon with exactly one year to maturity pays 4 in 6 months and 104 in 12 months. It is thus worth

$$0.98 \times 4 + 0.95 \times 104 = 102.72.$$

It should be clear: If we know the discount function $d(t)$, we can price bonds...

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Computing the Discount Function from bond Prices

We will consider a simple enough method for computing the discount function. We do this using a real world example.

Consider the following bond prices:

Table: Bond Prices on July 15 2008

CUSIP	MATURITY	CPN	NXT_CPN_DT	PX_DIRTY_ASK
912828EC0	8/15/2008	4.125	8/15/2008	101.9455701
912828EV8	2/15/2009	4.5	8/15/2008	103.2730082
912828CS7	8/15/2009	3.5	8/15/2008	102.8112981
912828DL1	2/15/2010	3.5	8/15/2008	103.3425481
9128276J6	8/15/2010	5.75	8/15/2008	109.0103022
9128276T4	2/15/2011	5	8/15/2008	108.0741758

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- Note that these are all semi annual coupon bonds with February/August coupon and maturity payments.
- Assume for now that we can trade at the asking price.
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Step 1: Finding $d(0.0833)$.

The first bond matures at 8/15/2008 with coupon rate 4.125% and thus has $30/360=0.0833$ years to expiration. It pays $100+4.125/2 = 102.0625$ on 8/15/2008.

It must be that

$$101.9455701 = d(0.0833)102.0625,$$

or

$$d(0.0833) = \frac{101.9455701}{102.0625} = 0.998854.$$

Thus, we have managed to figure out what $d(0.0833)$ is....

Step 2: Finding $d(0.5833)$

The next step is to find the value of a dollar paid on 2/15/2009. This is in seven months from July 15, 2008. ($7/12=0.5833$)

To figure out what $d(0.5833)$ is, we consider the 2/15/2009 maturity bond.

The 2/15/2009 bond pays 4.5% semi annual coupon.

The first coupon payment is on 8/15/2008 when it pays $4.5/2=2.25$. The present value of this coupon is

$$2.25 \times d(0.0883) = 2.25 \times 0.998854 = 2.247422242.$$

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The market value of the Feb 09 bond is 103.2730082 and it must satisfy

$$103.2730082 = 2.25 \times d(0.0833) + 102.25 \times d(0.5833)$$

Since $d(0.0833) = 0.998854$ we have that

$$103.2730082 = 2.247422242 + 102.25 \times d(0.5833)$$

or

$$d(0.5833) = \frac{103.2730082 - 2.247422242}{102.25} = 0.988025291.$$

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The remaining ones...

We similarly find that

$$d(1.08333) = \frac{\textit{Price} - .5 * \textit{coupon} \times S}{100 + .5 \times \textit{coupon}} \quad (4)$$

where S is the sum of all prior values of the discount function (in this case $S = 0.998854 + 0.988025$).

We get $d(1.0833) = 0.976258071$.

For $t = 1.5833$, we find

$$S = 0.998854 + 0.988025 + 0.976258 = 2.963137692$$

so that

$$d(1.5833) = \frac{103.3425481 - \frac{3.5}{2} \times 2.963137692}{100 + \frac{3.5}{2}} = 0.964688522$$

Similarly, we find

$$d(2.0833) = 0.949869277, d(2.5833) = 0.93541402, \text{ etc.}$$

The next figure plots the entire discount function computed from both bids and asks on July 15, 2008.

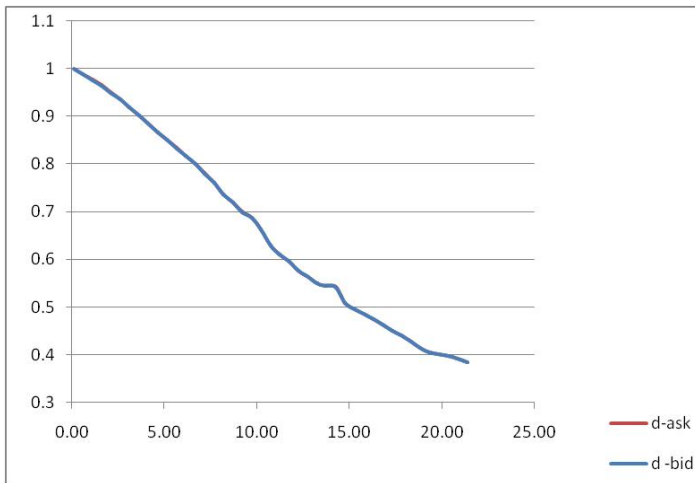


Figure: Discount function computed from bids and asks using FEB/AUG maturities on Jul 15, 2008

Comments on the discount function:

- Function computed from both asks and bids are visually indistinguishable
- Notable humps in the 15-17 yr range. Humps could be due to micro market structure effects. They COULD imply possible arbitrage opportunities (more on that later...)
- Notice that the 15-17 year range consists of old, off-the-run 30 years
- Curve flattens out at the end

Re-visiting zero coupons

Remember from last time: zero coupon bonds pay principal only.

Consider a zero coupon with maturity t and with \$1 principal. What is the current value?

It is just $d(t)$.

The discount function is identical to prices of zero coupon bonds if such exist (otherwise arb)

The *yield to maturity*, y_t of the zero coupon bond solves

$$d(t) = (1 + y(t))^{-t},$$

such that

$$y(t) = d(t)^{-\frac{1}{t}} - 1$$

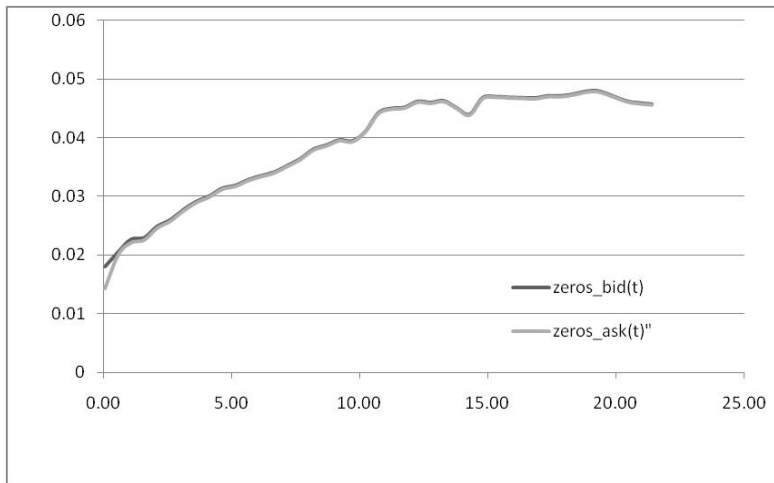


Figure: Zero coupon yield curve implied by bids and asks using FEB/AUG maturities on Jul 15, 2008

The spot rate function

Consider any point on the discount function, for example $d(0.5833)$. This is the value on 7/15/2008 of one dollar paid on 2/15/2009. This is the value *implied* by the prices of coupon bonds. The quantity

$$1/d(0.5833) - 1 = 1.1875\%$$

is the rate of return on the 7 month investment implied by the discount function.

The number

$$r(t) = 2(d(t))^{-\frac{1}{t2}} - 1$$

is the *annualized spot-rate implied by the discount function*.

The spot rate $r(t)$ is a semi-annually compounded yield curve associated with zero coupon claims.

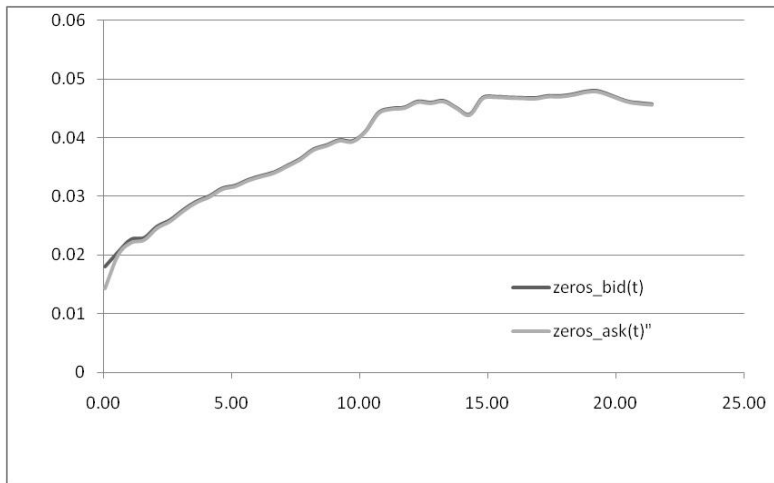


Figure: Spot rates implied by bids and asks using FEB/AUG maturities on Jul 15, 2008

Comments on the spot rate function:

- The spot rate function is in principle the same as the zero-coupon yield curve: It give the theoretical value of one dollar at some maturity t
- The previous spot rate curve is reasonably smooth in the 0-10 and 20-30 maturities, and relatively bumpy in the 10-20.
- It is a fairly typical zero coupon yield curve with an *upward slope*
- It is somewhat less common to find the *dip in the end*
- We will compare to zero coupon curves constructed using different methods later..
- We *could have* also computed a spot rate function using NOV/MAY maturities (see upcoming homework)

Shortcomings of the sequential method

- Does not work when there are no quotes for a particular maturity
- Requires all prices to be accurate. If not, $d(t)$ is not accurate either. For example, stale quotes for off-the-run 30 years may affect computations for the 10-20 year range.
- Yield curve looks strange with bumps and kinks. Could actually imply *negative forward* rates (see below) which again implies arbitrage.
- Almost certainly misprices NOV/MAY maturities.

Forward Rate Curve

Suppose there exists a forward contract such that we can enter into an agreement for a deliver of a T maturity zero coupon with a \$1 face in year $T - 1$.

Let $F(T)$ denote the forward price.

By entering into the forward contract, you receive the following cash flows (CF):

- At time 0: $CF=0$
- At time $T-1$: $CF=-F$ (we pay the forward and receive the bond)
- At time T : $CF=1$ (bond pays principal)

The forward contract implies a cash outflow of $F(T)$ (the forward price) at time $T - 1$ and gives \$ 1 at time T . In other words, by entering the forward contract, the investor *locks in* a risk free return of

$$f(T) = 1/F(T) - 1.$$

We can show by an absence of arbitrage argument that

$$F(T) = \frac{d(T)}{d(T-1)}$$

implying that

$$f(T) = \frac{d(T-1)}{d(T)} - 1$$

is the forward rate.

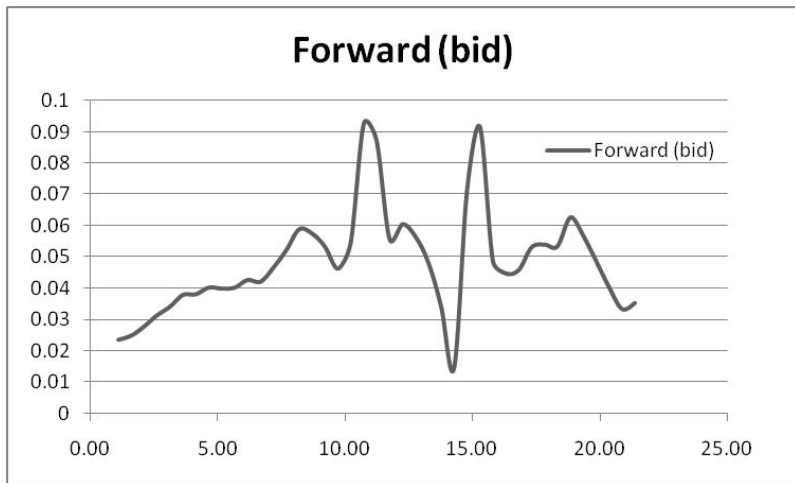


Figure: Forward rates implied by bids using FEB/AUG maturities on Jul 15, 2008

Features of the forward rate curve

- Many humps and bumps.
- Rates are positive
- Bumps are accentuated by bumps in the estimated zero curve
- Example: from Feb 2024-2025 forward rate is 1.8%. From Feb 2025-2026 we get 7.8%. Does this make sense?

Next time: Curve fitting allows us to estimate zero coupon curves that have less humps and bumps...