

# Hedging Interest Rate Risk I: One Factor Risk Measures

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# Bond Prices and Yields

- Consider a 2.625% coupon bond with maturity 3/15/2009. Suppose the bond is selling at with settlement 9/23/2008 at a price 100.534 implying a yield (to maturity) of 1.634%.
- We are interested in what would happen to the value of this bond if interest rates change.
- Suppose the ytm changes from 1.634 to 1.734 (a 10 basis point change). What is the impact on the price?

The price at  $y = 1.734$  is 100.487 - a -0.047 dollar change. Let  $\Delta P$  and  $\Delta y$  denote the change in the price and ytm, respectively.

We have  $\Delta P = -0.047$  given  $\Delta y = 0.1$ . The relative change is

$$\frac{\Delta P}{\Delta y} = \frac{-0.047}{0.1} = -0.47$$

(the number is -0.4724 computed to four digits)

Let's see what happens if  $y$  changes by only 5 basis points. Then the new price is 100.510 and

$$\Delta P = 100.510 - 100.534 = -0.024.$$

The relative change is

$$\frac{\Delta P}{\Delta y} = \frac{-0.024}{0.05} = -0.4725$$

Note that this number is close to the relative change when  $\Delta y = 0.1$ .

Lets try even smaller  $\Delta y$ 's: We get

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$y(\text{new})$	$P$	$\Delta P$	$\Delta y$	$\Delta P/\Delta y$
1.634001	100.534	0.000	1E-06	-0.4726
1.635	100.534	0.000	0.001	-0.4726
1.644	100.529	-0.005	0.01	-0.4726

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Note that as we change  $y$  very little, the ratio  $\Delta P/\Delta y$  approaches the number -0.4726.

This number is the *derivative* of the price as a function of the ytm at the point  $y = 1.634$ . The derivative is denoted

$$\frac{dP}{dy}$$

We have found

$$\frac{dP}{dy} = -0.4726$$

at  $y = 1.634$ .

It is easy to demonstrate that a different price/ yield combination will imply a different derivative. For example if  $y = 5.634$  then  $dP/dy = -0.4592$ .

The risk measure “DV01” measures the sensitivity of a bond’s price to a 0.01 change in the “interest rate,” typically measured by the ytm of the bond.

DV01 gives the approximate absolute change in the value of the bond price in response to a 1 basis point change in the yield divided by 100,

$$DV01 = -\frac{\Delta P}{100\Delta y}$$

with  $\Delta y = 0.01$ .

I.e, a change from 5% to 5.01%.

Note: Tuckman defines

$$DV01 = -\frac{\Delta P}{10000\Delta y}$$

The difference is due to defining  $y$  in percentage (here) and decimal (Tuckman).

I,e, a change from 0.05 to 0.0501.

The dollar duration of a bond is by definition

$$D_{\$} = -\frac{dP}{dy}$$

it differs by a factor of 100 from DV01 (as defined here).

The derivative of simple functions such as polynomials can often be found explicitly. For example,

$$\frac{dx^2}{dx} = 2x$$

and

$$\frac{d(1+y)^{-T}}{dy} = -T(1+y)^{-T-1}$$

etc.

In our example  $D_{\$}$  is just 0.4726.

Bloomberg reports it to 0.474. Why different? Hard to say... most likely they are using different ytm/date conventions and/or they could have computed it from an old bid...

We will consider the call option example in Tuckman. In this example, a market maker sells 100 M worth of face value of a call option on a bond.

- The bond is selling at par.
- The DV01 of the option at  $y=5\%$  is 0.0369.
- The DV01 of the bond at  $y=5\%$  is 0.0779.
- In other words, the option position will change by \$ 0.0369 for every "small change" in the ytm on the bond. The bond changes by \$ 0.0779 at the same time.

Intuitively, you probably can guess that we'd like to delta hedge this option by buying  $0.0369/0.0779=0.4737$  bonds per option contract. That is, we buy 47.37 M worth of bonds.

Indeed, the correct number to hedge is

$$F_a = \frac{-F_b DV01_b}{DV01_a} \quad (1)$$

where

- $F_a, F_b$  is the face amount of securities  $a$  and  $b$ .
- $DV01_a$  and  $DV01_b$  are the DV01's of securities  $a$  and  $b$ .

# Does it work...?

The value of the position is

$$-100,000,000 \times \frac{3.0501}{100} + 47,370,000 \times \frac{100}{100} = 44,319,900.$$

Now suppose the ytm changes to 4.99%. The value of the option is now 3.0871, the bond 100.078 and the total position is hence

$$-100,000,000 \times \frac{3.0871}{100} + 47,370,000 \times \frac{100.078}{100} = 44,319,849$$

.... so the hedge works almost perfectly.

What if the ytm increases to 5.01?

Then the option is worth 3.0134, the bond is worth 99.9221 and so the total position is

$$-100,000,000 \times \frac{3.0134}{100} + 47,370,000 \times \frac{99.9221}{100} = 44,319,698,$$

a net loss of 202.

Note: Option position loses

$100,000,000 \times (3.0871 - 3.0501)/100 = 3,700$  if not hedged.

# Macauley's Duration

Suppose we are in a simple world where future coupon payments are received in 6,12,.. months from now. If so,

$$P = \sum_{t=1}^{2T} \frac{C_t}{(1 + y/2)^t}$$

is the price of the bond.

Now define

$$D = \sum_{t=1}^{2T} tw_t$$

where

$$w_t = \frac{1}{P} \frac{C_t}{(1 + y/2)^t}$$

since  $\sum_t w_t = 1$  we can interpret  $D$  as a *weighted average* of future cash flow times. The weights are proportional to the present value of the bond's cash flows.

is defined as

$$D^* = \frac{D}{1 + y/2}$$

using semi-annual compounding, and

$$D^* = \frac{D}{1 + y}$$

using annual compounding.

# Key duration relationships

The following identities obtain

$$D_{\$} = -\frac{dP}{dy} \quad (2)$$

$$D = -\frac{(1 + y/2)}{P} \times \frac{dP}{dy} = \frac{(1 + y/2)}{P} D_{\$} \quad (3)$$

$$D^{*} = \frac{D}{1 + y/2} = \frac{1}{P} D_{\$} \quad (4)$$

assuming semi annual compounding.

# Some Bloomberg Risk Numbers

CUSIP	MATUR	CPN	BID	$D_{\S}$	D	$D^*$
912828DP2	3/15/2010	4	102.937	1.476	1.446	1.432
912828HU7	3/31/2010	1.75	99.5	1.480	1.49	1.475
912828DR8	4/15/2010	4	103.062	1.558	1.501	1.486
912828HX1	4/30/2010	2.125	101.006	1.56	1.5699	1.554
912828DU1	5/15/2010	3.875	103.046	1.64	1.5860	1.571
912810EF1	5/15/2020	8.75	141.703	11.433	8.058	7.893
912810EG9	8/15/2020	8.75	142.915	11.627	8.303	8.132

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Can see that the duration and modified duration relate to dollar duration as follows (take 8/15/2020):

$$D = 11.672 \times \frac{1 + 0.04218/2}{142.031} = 8.303$$

which is identical to the 8.302 reported by Bloomberg.

$$D^* = \frac{8.303}{1 + 0.04218/2} = 8.132$$

which is again identical.

# Duration of zeros

The price of a zero with a \$ 1 face is

$$P = (1 + y)^{-T}.$$

Hence, we can compute the dollar duration as

$$D_{\$} = -\frac{dP}{dy} = T(1 + y)^{-T-1}.$$

The Macauley duration is

$$D = \frac{(1 + y)}{P} D_{\$} = T(1 + y)^{-T} / P = T.$$

Thus, the duration is literally the maturity of zero coupon bonds.

The convexity measures how much the duration changes with respect to a change in the ytm. Mathematically,

$$C = \frac{1}{P} \frac{d^2 P}{dy^2}$$

- the second derivative of the price wrt the yield, scaled by the price.

Convexity can be used to predict changes in prices more accurately than we can using only the duration. We will not ever do this. It is a waste of time.

# Portfolio Duration

Consider a portfolio of  $N$  different interest rate sensitive assets.  
Let  $n_i$  be the number of asset  $i$  held.

Then

$$\sum_{i=1}^N n_i P_i$$

is the portfolio value. We will claim without proof that

$$\sum_{i=1}^N n_i D_{\$,i}$$

is the portfolio (dollar) duration.

- Lets consider a portfolio where we are short one 8.75 of 8/2020 worth 142.931 at a ytm of 4.218.
- Lets hedge this using the 2.125 of 4/30/2010. This bond has price 101.006 at a ytm of 2.025. The dollar durations (Bloomberg) are 11.627 and 1.56 respectively
- We need to buy

$$-\frac{11.627}{1.56} = 7.4532$$

units the shorter maturity to obtain a zero dollar duration portfolio.

Lets see how our duration neutral portfolio holds up when rates change.

First, lets assume the ytm's increase by 10 basis points to 4.318 and 2.125, respectively.

The new prices are 141.757 and 100.849 respectively.

The total P&L from this trade is

$$\begin{aligned} & -1 \times (141.757 - 142.915) + 7.4532 \times (100.692 - 101.006) \\ & = 1.1575 - 1.168 \approx -0.01 \quad (5) \end{aligned}$$

Next, consider what would happen if short rates increase *more* than long rates which is typically what happens when the Fed change short term rates. So lets assume 20 and 10 basis point drops.

The total P&L from this trade is now \$ -1.18.

# What happened?

Zero duration portfolios are only immune to *parallel* yield curve shifts.

Thus, they are in fact very risky.  
Would hedging convexity help?

We'll think about how to deal with this next time. One way to think about is is that if we know that when the short rate change by, say  $\Delta y_{\text{short}}$  then long rates typically change by  $\Delta y_{\text{long}} = x \times \Delta y_{\text{short}}$  then we could perhaps be successful if we found a good number for  $x$ ..