Interest Rate Trees

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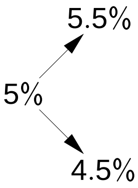
- Suppose we have a simple economy where *spot* rates can go either up or down by a fixed amount over the next six months.
- Remember: the spot rate is the semi-annually computed ytm on a zero.
- For example :



Lets consider a bond with $ 1000 face with a six month maturity. At time 0 it is worth

$$\frac{1000}{1 + \frac{0.05}{2}} = \$975.61$$

If, on the other hand, we are in the "up-state six months from now, a six month bond is worth

$$\frac{1000}{1 + \frac{0.055}{2}} = \$973.24$$

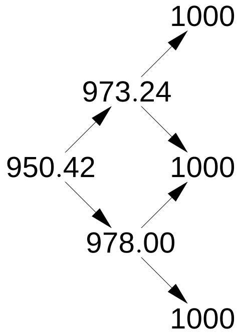
Of course, in the down state we have

$$\frac{1000}{1 + \frac{0.045}{2}} = \$978$$

Moreover, if we assume that the *one year* spot rate is 5.15%, we get the time zero price of the one year zero as

$$\frac{1000}{(1+\frac{0.0515}{2})^2} = \$950.42$$

We have the following tree for the price of the one year



Risk-Neutral Probabilities

- Suppose we were to assume that the probability of an up/down move were 50/50.
- In this case, we should simply compute the value of the one year bond at time zero as the expected future value of the bond, discounted at the risk free rate.
- If so, we get

$$\frac{\frac{1}{2}973.24 + \frac{1}{2}978}{1 + 0.05/2} = 951.82$$

which is different from the actual price of 950.42.

What is going on? Clearly, the market prices are inconsistent with the probability of $1/2$ for an up/down move.
Let's see what the probability of an up/down move must be....
Let p denote the probability of an up-move, $1 - p$ a down move
It must be that

$$\frac{p \times 973.24 + (1 - p) \times 978}{1 + 0.05/2} = 950.42$$

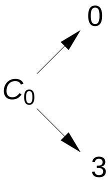
Solving for p gives $p = 0.8024$.

About Risk Neutral Probability Measures

- Risk neutral probabilities may not equal the actual probability of an up down move (called the "objective" measure)
- Risk neutral probabilities are implied *by absence of arbitrage*
- Notice that we obtained the risk neutral probabilities simply from knowing the spot rates and the possible future interest rate path
- We can interpret risk neutral probabilities as *risk adjusted* probabilities
- Complete markets = all claims can be hedged.
- The fundamental theorem of asset pricing: If markets are *complete* and there are no arbitrage opportunities, there exists a unique risk neutral probability measure. If there exists a unique risk neutral probability measure, markets are complete and there are no arbitrage opportunities.

Valuing derivatives

Consider an option on a bond with six month maturity to buy the one year zero at a strike of $ 975.
Its value is given in the tree



To find the value of the option at time zero we simply use compute the expected payoff under the risk neutral measure

$$C_0 = \frac{0.8024 \times 0 + (1 - 0.8024) \times 3}{1 + 0.05/2} = \$0.58$$

Hedging

Lets see if we can design a portfolio of the 6 and 12 month bonds to replicate the payoff on the call option.
Let F_6 and F_{12} denote the positions in these bonds. We want the payoffs of the replicating portfolio to equal the option in each state:

$$F_6 1000 + F_{12} 973.24 = 0 \quad (1)$$
$$F_6 1000 + F_{12} 978 = 3 \quad (2)$$

- a system of two equations in two unknowns. Subtracting (1) from (2) gives

$$F_{12}(978 - 973.24) = 3$$

so $F_{12} = 0.6302521$ and
$F_6 = -973.24 \times 0.6302521/1000 = -0.61338655$

Thus we get,

$$-0.61338655 * 1000 + 973.24 * 0.6302521 = 0$$

in the up state, and

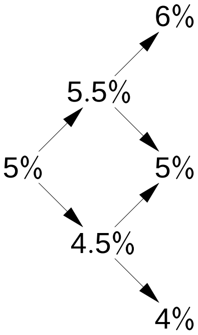
$$-0.61338655 * 1000 + 978 * 0.6302521 = 3$$

in the down state.
We can now verify the price of the option. It must equal the value of the replicating portfolio

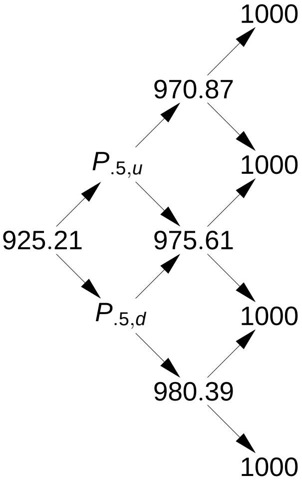
$$-0.61338655 * 975.61 + 950.42 * 0.6302521 = 0.58$$

Multiple Periods

Suppose



Assume further that the 1.5 year spot rate is 5.25%. In this case we recover the price of a 1.5 year zero as



Here of course, we use the last nodes in the interest rate tree to compute the prices at time 1.5. For example

$$970.87 = \frac{1000}{1 + 0.06/2}$$

and so on.
Similarly, on date 0 the value of the bond is just

$$\frac{1000}{(1 + 0.0525/2)^3} = 925.21$$

But how do we find $P_{.5,u}$ and $P_{.5,d}$?? Let's see if we can recover the risk-neutral probabilities of an up/down move at time 0.5. We know that the risk neutral probability of an up move in the first period is $p = 0.8024$. This means that the following equation must hold:

$$\frac{0.8024P_{.5,u} + 0.1976P_{.5,d}}{1 + 0.05/2} = 925.21 \tag{3}$$

Now let q denote the risk neutral probability of an up-move at $t = 0.5$. It must be that

$$P_{.5,u} = \frac{970.87q + 975.61(1-q)}{1 + 0.055/2} \tag{4}$$

and

$$P_{.5,d} = \frac{975.61q + 980.39(1-q)}{1 + 0.045/2} \tag{5}$$

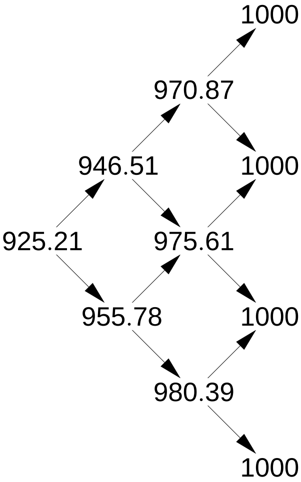
Let's substitute (4) and (5) into (6). This gives

$$\frac{0.8024\frac{970.87q+975.61(1-q)}{1+0.055/2}+0.1976\frac{975.61q+980.39(1-q)}{1+0.045/2}}{1+0.05/2}=925.21 \tag{6}$$

which we can now solve for q. We find

$$q = 0.6489$$

Thus



- Note: we recovered q by assuming that the probability of an up/down move would be the same in either state at time 0.5.
- We can also assume that the probability is *state-dependent*. In this case, we will have to use a different technique for building the tree.
- We also would like to make the tree using much smaller time steps. For example, we might use one, or two time-steps per day. With two time-steps per day, a thirty year tree will have $365 \times 30 \times 2 = 21,900$ nodes steps and some 239,837,851 nodes (239 M).

Application: Constant maturity treasury swaps

This contract pays the difference between the yield on some bond and a fixed rate times a notional amount. The book uses the example

$$\frac{1,000,000}{2}(y_{cmt} - 5\%)$$

where y_{cmt} is just the spot rate given in our interest rate tree. Thus, for the first period we get

$$\frac{1,000,000}{2}(5.5\% - 5\%) = 2,500$$

if rates go up, and

$$\frac{1,000,000}{2}(4.5\% - 5\%) = -2,500$$

if they go down.

Similarly, we find that three different states at time 1 produce payoffs,

$$\begin{aligned}
\frac{1,000,000}{2}(6\% - 5\%) &= 5,000 \\
\frac{1,000,000}{2}(5\% - 5\%) &= 0 \\
\frac{1,000,000}{2}(4\% - 5\%) &= -5,000
\end{aligned}$$

The value of the swap at time $t = 1/2$ in the up state is

$$\begin{aligned} & 2500 + \text{present value of expected future payoff} \\ = \; & 2,500 + \frac{0.6489 \times 5,000 + 0.3511 \times 0}{1 + .055/2} = 2,500 + 3,157.66 \\ = \; & 5,657.66 \end{aligned}$$

Similarly in the down state

$$\begin{aligned} & 2500 + \text{present value of expected future payoff} \\ = \; & -2,500 + \frac{0.6489 \times 0 + 0.3511 \times -5,000}{1 + .045/2} = -4,216.87 \end{aligned}$$

We now find the time 0 value of the swap as

$$\frac{0.8024 \times 5,657.66 + 0.1976 \times -4,216.87}{1 + 0.05/2} = 3,616.05$$

Note that all expectations are computed from the risk neutral probabilities - not the "objective" probabilities (1/2). If we had used the objective probabilities, the value of the swap would have been exactly..... ?

Concluding remarks

- Interest rate trees are simple devices for modeling the random movements in interest rates
- Interest rate trees are easy to adapt to pricing new derivative securities
- Famous models:
  - Original Salomon Bros model (no authors)
  - Ho-Lee
  - Black-Derman-Toy
  - Black-Karasinsky
  - Hull-White
- Shortcoming: Difficult to consider more than one source of randomness. For example, it is hard to make a model that allows for variation in both level and slope