

Two Factor Models

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Limitations of One Factor Models..

- Bond yields are 100% correlated.
- Limited term structure shapes
- No room for independent moves in level, slope (say)
- Let $R(\infty)$ denote the ytm on a zero with very long maturity (approaching infinity). This ytm is constant for one factor short rate models

Limitations of term structure shapes in Vasicek

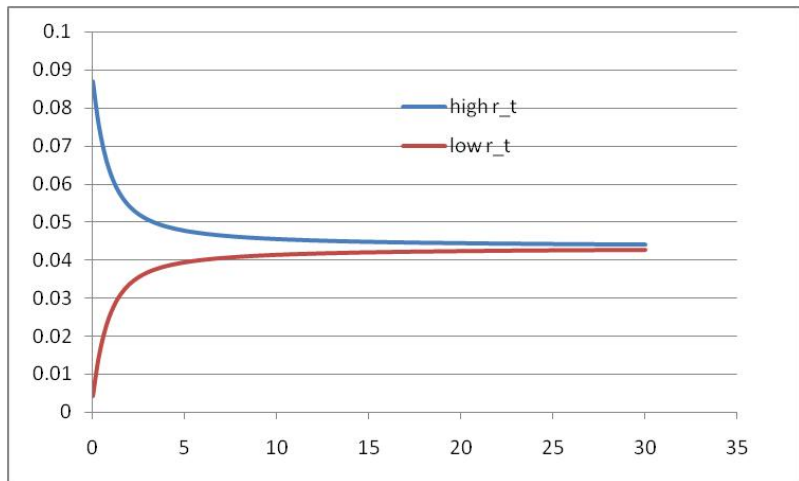


Figure: Vasicek yield curves.

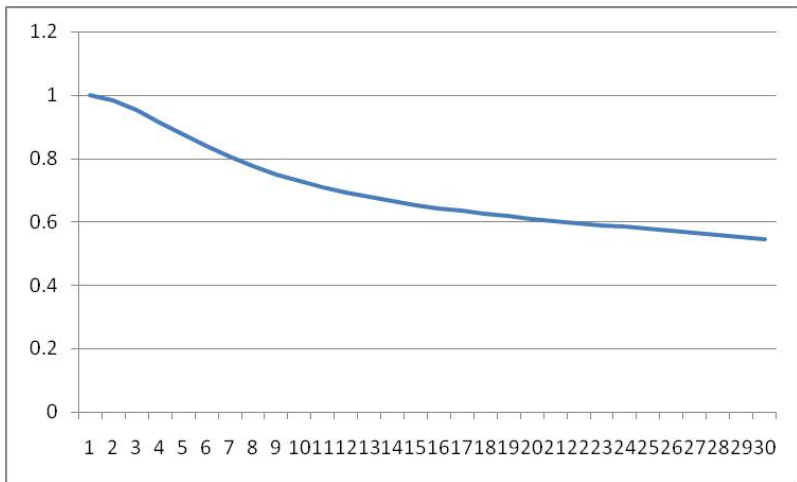


Figure: Correlations between one year ytm and ytm of other maturity bonds (x-axis).

Hedging in one factor models

Duration hedging almost works in one factor CIR and Vasicek models. This is so because yields are (almost) 100% correlated.

In other words, (in the Vasicek model) the yield on one bond moves in parallel to another, meaning that we can hedge a position on one bond with another, irrespective of maturity of the bonds.

This is clearly not true empirically...

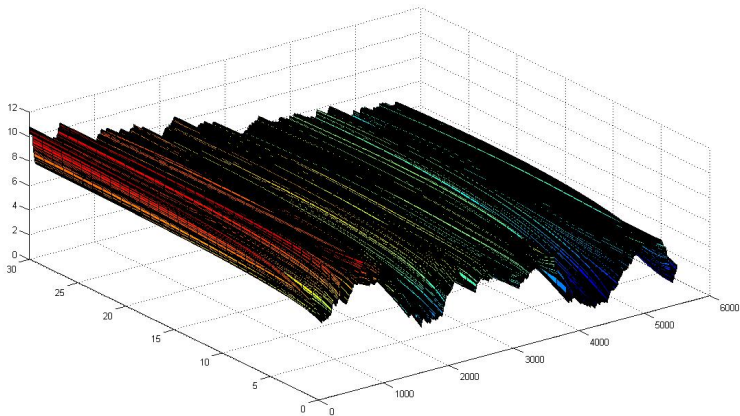


Figure: Historical zero coupon yield curves 85-08.

Problem with one factor models that fit the yield curve

Models such as Ho-Lee, Hull-White, Black-Derman-Toy and Black-Karasinsky make sure to *fit the initial yield curve*.

Of course, tomorrow we have a different (initial) yield curve...

The BDT and other models need to be re-calibrated in this case. The problem is that this re-calibration almost certainly implies changing the fundamental probability model. This leads to a *time* inconsistency in the approach...

Basic idea of two and n factor models

- When yields are driven by two factors they are no longer perfectly correlated.
- Gives rise to various different shapes of yield curves
- Gives us different ways to think about hedging (need more than one bond to hedge...)

An example model

Let the short rate, r_t , be the sum of two factors x_t and y_t ,

$$r_t = x_t + y_t$$

We now assume *different* statistical behavior for x_t and y_t . In particular, we will just assume that both follow an AR(1) like process

$$dx_t = \kappa_x(\theta_x - x_t)dt + \sigma_x dW_t^x \quad (1)$$

$$dy_t = \kappa_y(\theta_y - y_t)dt + \sigma_y dW_t^y \quad (2)$$

This is a *two-factor Vasicek* model.

- κ_x and κ_y : speed of mean reversion for factors x and y respectively.
- θ_x and θ_y : long run average for factors x and y respectively.
- σ_x and σ_y : volatility for factors x and y respectively.
- dW_t^x and dW_t^y : random shocks to factors x and y respectively.
- dW^x and dW^y can be correlated with correlation coefficient ρ .

The spot rate for the two factor Vasicek model is

$$\begin{aligned} \hat{r}(T) = & \theta_x + \frac{1 - e^{-\kappa_x T}}{\kappa_x T} (x_0 - \theta_x) + \theta_y + \frac{1 - e^{-\kappa_y T}}{\kappa_y T} (y_0 - \theta_y) \\ & - \frac{\sigma_x^2}{2\kappa_x^2} \left(1 + \frac{1 - e^{-2\kappa_x T}}{2\kappa_x T} - 2 \frac{1 - e^{-\kappa_x T}}{\kappa_x T} \right) \\ & - \frac{\sigma_y^2}{2\kappa_y^2} \left(1 + \frac{1 - e^{-2\kappa_y T}}{2\kappa_y T} - 2 \frac{1 - e^{-\kappa_y T}}{\kappa_y T} \right) \\ & - \frac{\rho\sigma_x\sigma_y}{\kappa_x\kappa_y} \left(1 - \frac{1 - e^{-\kappa_x T}}{2\kappa_x T} - \frac{1 - e^{-2\kappa_y T}}{2\kappa_y T} + \frac{1 - e^{-(\kappa_x + \kappa_y)T}}{(\kappa_x + \kappa_y)T} \right). \end{aligned} \quad (3)$$

Lets look at some different yield curves....

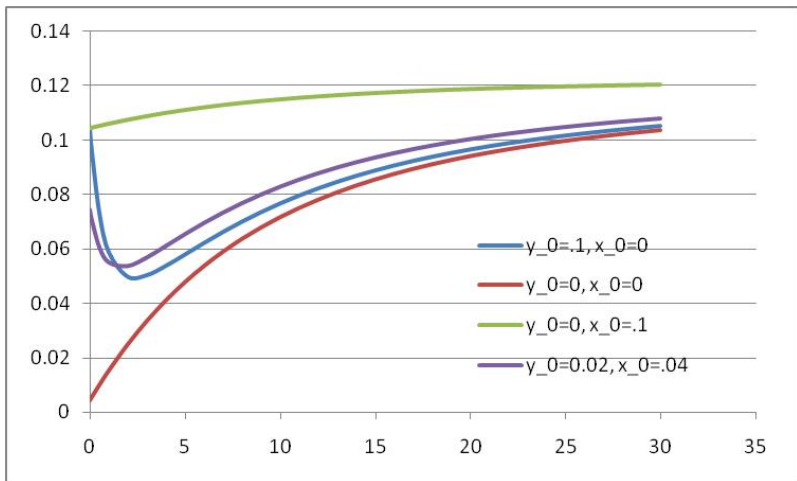


Figure: Two Factor Vasicek yield curves.

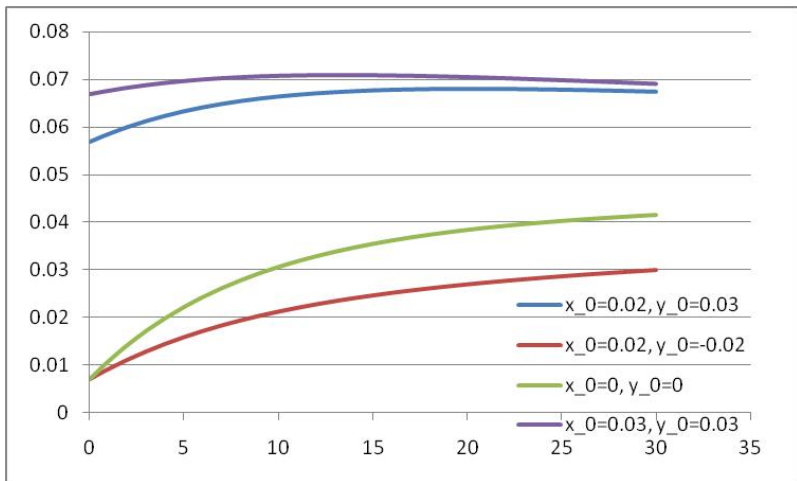


Figure: Two Factor Vasicek yield curves. Different parameters.

See that the two factor model is capable of generating a wide range of different term structures...

- Hump shapes
- Different levels for short and long rates
- Still smooth....

Implementing Two Factor models

One primary problem with multi-factor models is that we typically do not observe the factors, x and y .

How do we deal with this?

Lets take two zero coupon bonds with maturities T_1 and T_2 . We generally cannot observe the two factors x and y . However, the yields, $Y_1(t)$ and $Y_2(t)$ of the two bonds are related to the unobserved factors

$$Y_1(t) = A_1 + B_{1,x}x(t) + B_{1,y}y(t) \quad (4)$$

$$Y_2(t) = A_2 + B_{2,x}x(t) + B_{2,y}y(t) \quad (5)$$

where $A_1, A_2, B_{1,x}, B_{1,y}, B_{2,x}$ and $B_{2,y}$ can be recovered from the expression for the spot rate. These coefficients depend on the maturity of the bond as well as the parameters $\kappa_x, \kappa_y, ..$ etc.

For example

$$B_{1,x} = \frac{1 - e^{-\kappa_x T_1}}{\kappa_x T_1}$$

and

$$B_{2,y} = \frac{1 - e^{-\kappa_y T_2}}{\kappa_y T_2}$$

and so on.

We can now solve (4) and (5) for $x(t)$ and $x(t)$ using the standard technique for solving two linear equations in two unknowns.

Building two factor trees

Building multidimensional trees is complicated. In case of a two factor model, we can try to visualize a three-dimensional tree (just like the green ones outside), growing from the ground up in two dimensions. In our case the height of the tree represents time, while east/west/north/south dimension represents the two factors.

We cannot build such trees easily in Excel because spreadsheets are fundamentally two dimensional representations of data (rows \times columns)

Typically, three dimensional trees should be implemented using specialized computing environments. For example, a *three dimensional array* is an object where

$$X(i, j, k)$$

represents the i, j and k indexed elements.

Illustrating a three dimensional tree

$$(x^{dd}, y^{uu}) \quad (x_0, y^{uu}) \quad (x^{uu}, y^{uu})$$

$$(x^d, y^u) \quad (x^u, y^u)$$

$$(x^{dd}, y_0) \quad (x_0, y_0) \quad (x^{uu}, y_0)$$

$$(x^d, y^d) \quad (x^u, y^d)$$

$$(x^{dd}, y^{dd}) \quad (x_0, y^{dd}) \quad (x^{uu}, y^{dd})$$

For this tree, we now have the following possibilities for the short rate:

At time 1 the 4 states closest to the initial states are possible.

At time 2, there are 13 possible states.

At time n , there are $\sum_i^{n+1} i^2$ giving 2,870 nodes for $n = 20$ time steps and 3.3M for $n=100$.

The approach is.

- 1 Build a tree for x and y separately.
- 2 Calibrate probabilities of the various branches to match the correlation

Specifics are beyond the scope of this lecture. Toy example is given in Tuckman's book. Note that the Tuckman procedure is not robust to negative probabilities.

Concluding Remarks

- We can derive mult-factor CIR models as well
- Vasicek and CIR are subsets of the popular "affine" class of models.
- Affine, Vasicek and CIR models are produce analytical pricing formulas for a wide range of derivatives, including options.
- Analytical results are not available for path dependent derivatives. Path-dependency = the value at expiration depends not only on the state of the economy at that point, but also previous states (i.e., history matters). Example of simple path dependent claim: American puts